Resolution of the time dependent Pn equations by a Godunov type scheme having the diffusion limit. Part 1 : The case of the time dependent P1 equations

Patricia Cargo, Gerald Samba

Resolution of the time dependent P_n equations by a Godunov type scheme having the diffusion limit. Part 1 : The case of the time dependent P_1 equations.

P.Cargo and G.Samba
Commissariat à l'Energie Atomique,
Bruyères Le Châtel, 91297 Arpajon Cedex, France

11th February 2008

1 Introduction

The P_n equations are a good tool to approximate the neutral transport equation. This model does not suffer from ray effects known to seriously affect the accuracy of the discrete ordinates method S_n [4, 5].

Recently [6, 7, 8, 9], implicit Godunov type schemes have been used to approximate the P_n model in one dimension since the resulting system is hyperbolic. The advantage of such methods is to take exactly into account the wave structure involved. The main issue is the treatment of the collision terms. If no care is taken, the scheme is unable to treat diffusive medium with a mesh whose cell size is much larger than the mean free path. In this paper, we adress this issue in 1D. Let us now explain what it means for the P_1 equations:

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \dfrac{1}{\sqrt{3}} \dfrac{\partial J}{\partial x} + \sigma_a \rho = 0 \\ \dfrac{\partial J}{\partial t} + \dfrac{1}{\sqrt{3}} \dfrac{\partial \rho}{\partial x} + \sigma_t J = 0 \end{cases} \tag{1}$$

where ρ and J denote the two first moments of the solution $\psi(x, \mu, t)$ of the transport equation:

$$\frac{\partial \psi}{\partial t} + \mu \frac{\partial \psi}{\partial x} + \sigma_t \psi = (\sigma_t - \sigma_a) \tilde{\psi} \tag{2}$$

with $\mu \in [-1, +1]$, $x \in [x_m, x_M]$, $\tilde{\psi} = \dfrac{1}{2} \displaystyle\int_{-1}^{+1} \psi d\mu$. The coefficient σ_a denotes the absorption cross section and σ_t the total cross section. Note that the speed of the neutrons does not appear in (1) because the transport equation (2) has been adimensioned.

In a diffusive regime, the system (1) may be scaled like:

$$\frac{\partial}{\partial t} \mapsto \epsilon \frac{\partial}{\partial t}, \quad \sigma_t \mapsto \frac{\sigma_t}{\epsilon}, \quad \sigma_a \mapsto \epsilon \sigma_a.$$

This scaling leads to the following equations:

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \dfrac{1}{\epsilon\sqrt{3}}\dfrac{\partial J}{\partial x} + \sigma_a \rho = 0 \\ \dfrac{\partial J}{\partial t} + \dfrac{1}{\epsilon\sqrt{3}}\dfrac{\partial \rho}{\partial x} + \dfrac{\sigma_t}{\epsilon^2} J = 0. \end{cases} \tag{3}$$

It can be shown that when ϵ goes to zero, ρ converges towards the solution of the diffusion equation:

$$\frac{\partial \rho}{\partial t} - \frac{\partial}{\partial x}\left(\frac{1}{3\sigma_t}\frac{\partial \rho}{\partial x}\right) + \sigma_a \rho \quad = 0. \tag{4}$$

Unless the mesh is at the mean free path scale $\dfrac{\epsilon}{\sigma_t}$, an explicit or splitted treatment of the source terms $\left\{\sigma_a \rho, \dfrac{\sigma_t}{\epsilon^2} J\right\}$ does not give a correct approximation of ρ solution of (4) when ϵ goes to zero. To discretize the P_n model, the authors [7] propose a Godunov scheme where the dissipation is scaled. This scheme gives a good approximation of the solution of (4). But in a diffusive regime, it results in two uncoupled discretizations subject to non physical modes: one of them involves odd cells and the other one even cells.

In [8, 9], L. Gosse and al. have considered the Goldstein-Taylor model, also known as the telegraph equation:

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \dfrac{\partial J}{\partial x} = 0 \\ \dfrac{\partial J}{\partial t} + \dfrac{\partial \rho}{\partial x} + \sigma_t J = 0. \end{cases} \tag{5}$$

Following [10], they proposed a formulation of these equations which allows the exact resolution of the Riemann problem. The resulting Godunov scheme preserves the steady solutions of the system (5) (well-balanced property [10]). Moreover, this scheme gives the solution of the diffusion equation on a mesh at the diffusion scale $\sqrt{\dfrac{T}{\sigma_t}}$ not depending on ϵ (T is a characteristic time of evolution of the solution).

In [2, 3], a relaxation method applied to a two moments method for radiation leads to two systems close to (5). Their discretization is achieved by the scheme proposed in [8].

In this paper, before solving the P_n equations, we propose to extend the scheme of L. Gosse to the case of the P_1 model.

This article is organized as follows. We begin by presenting the main features of the P_1 model, among them its equivalence with the S_2 equations. The next section is devoted to the discretization of the P_1 equations: we propose a Gosse type scheme. It is proved that this scheme has the diffusion limit. We describe also how to deal with a variable size mesh and non constant coefficients (σ_a, σ_t). In the next section, some numerical results are presented. The last section is devoted to the conclusion.

2 P_1 equations and S_2 equations

We first show the equivalence between the P_1 equations and the so called S_2 equations in 1D. The P_1 equations are an approximation of the transport equation in a sense we are going to explain.

We are looking for the solution $\psi(x,\mu,t)$ of the transport equation (2) in plane geometry, with the boundary conditions:

$$\begin{cases} \psi(x_m,\mu,t) = \psi_m(\mu,t) \text{ for } \mu > 0 \\ \psi(x_M,\mu,t) = \psi_M(\mu,t) \text{ for } \mu < 0. \end{cases}$$

To derive the S_2 equations, we approximate $\psi(x,\mu,t)$ at two values of μ, $\mu^\pm = \pm\frac{1}{\sqrt{3}}$ and $\tilde{\psi}$ by the quadrature formula $0.5(\psi(x,\mu^-,t)+\psi(x,\mu^+,t))$. Let us denote:

$$\begin{cases} u = \psi(x,\mu^+,t) \\ v = \psi(x,\mu^-,t) \end{cases}$$

we obtain the S_2 equations:

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{1}{\sqrt{3}}\dfrac{\partial u}{\partial x} + \sigma_t u = \dfrac{1}{2}(\sigma_t - \sigma_a)(u+v) \\ \dfrac{\partial v}{\partial t} - \dfrac{1}{\sqrt{3}}\dfrac{\partial v}{\partial x} + \sigma_t v = \dfrac{1}{2}(\sigma_t - \sigma_a)(u+v) \end{cases} \tag{6}$$

with the boundary conditions:

$$\begin{cases} u(x_m,t) = \psi_m(\mu^+,t) \\ v(x_M,t) = \psi_M(\mu^-,t). \end{cases}$$

If we rewrite the system (6) as follows:

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{1}{\sqrt{3}}\dfrac{\partial u}{\partial x} = \dfrac{\sigma_t}{2}(v-u) - \dfrac{\sigma_a}{2}(u+v) \\ \dfrac{\partial v}{\partial t} - \dfrac{1}{\sqrt{3}}\dfrac{\partial v}{\partial x} = \dfrac{\sigma_t}{2}(u-v) - \dfrac{\sigma_a}{2}(u+v) \end{cases} \tag{7}$$

it is easy to check that:

$$\begin{cases} \rho = u+v \\ J = u-v \end{cases}$$

are solutions of the system (1) with the following boundary conditions :

$$\begin{cases} \left(\dfrac{\rho+J}{2}\right)(x_m,t) = \psi_m(\mu^+,t) \\ \left(\dfrac{\rho-J}{2}\right)(x_M,t) = \psi_M(\mu^-,t). \end{cases}$$

Note that u and v are decoupled in the left hand side of (7), but they are coupled in the right hand side. Actually, we have diagonalized the transport part of the hyperbolic P_1 system: the fields associated with the S_2 equations, (u,v), are the characteristic variables of the P_1 system, with the eigenvalues $\left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$.

3 Derivation of a Gosse type scheme

We consider a uniform mesh of size h to discretize the spatial domain $[x_m, x_M]$. The cells are defined by $C_j = [x_{j-1/2}, x_{j+1/2}]$ where $x_{j+1/2} = x_j + \dfrac{h}{2}$, $j \in [1, N_x]$.

3.1 Approximate Riemann solver

Following the ideas of L. Gosse and al. in [8], the terms on the right hand side of the system (7) are modified as follows:

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{1}{\sqrt{3}}\dfrac{\partial u}{\partial x} = h\displaystyle\sum_j \left(\frac{\sigma_t}{2}(v-u) - \frac{\sigma_a}{2}(u+v)\right)\delta(x - x_{j-1/2}) \\ \dfrac{\partial v}{\partial t} - \dfrac{1}{\sqrt{3}}\dfrac{\partial v}{\partial x} = h\displaystyle\sum_j \left(\frac{\sigma_t}{2}(u-v) - \frac{\sigma_a}{2}(u+v)\right)\delta(x - x_{j-1/2}) \end{cases} \tag{8}$$

where $\delta(x - x_0)$ stands for the Dirac mass in $x = x_0$.

The Riemann problem associated with (8) involves a stationary contact discontinuity, which yields two unknown states $\{\hat{u},\, \hat{v}\}$:

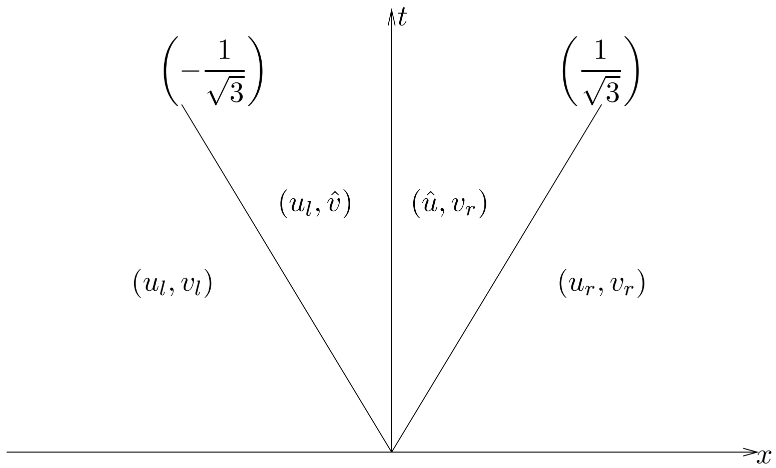

Figure 1: Riemann problem with the standing wave.

By analogy with [8]-[9], these states are computed by solving the steady equations:

$$\begin{cases} \dfrac{\partial u^*}{\partial \chi} = h\sqrt{3}\left(\dfrac{\sigma_t}{2}(v^* - u^*) - \dfrac{\sigma_a}{2}(u^* + v^*)\right) \\ -\dfrac{\partial v^*}{\partial \chi} = h\sqrt{3}\left(\dfrac{\sigma_t}{2}(u^* - v^*) - \dfrac{\sigma_a}{2}(u^* + v^*)\right) \end{cases} \quad \text{for } \chi \in [0,1] \tag{9}$$

for $\chi \in [0, 1]$ and the following boundary conditions:

$$\left\{\begin{array}{l} u^*(0) = u_l \\ v^*(1) = v_r. \end{array}\right. \tag{10}$$

By shifting to the (ρ^*, J^*) variables where $J^* = u^* - v^*$, $\rho^* = u^* + v^*$, we obtain:

$$\left\{\begin{array}{ll} \dfrac{\partial J^*}{\partial \chi} & = -2\beta\rho^* \\ \dfrac{\partial \rho^*}{\partial \chi} & = -2\alpha J^* \end{array}\right. \tag{11}$$

with $\alpha = h\sqrt{3}\dfrac{\sigma_t}{2}$, $\beta = h\sqrt{3}\dfrac{\sigma_a}{2}$ and the boundary conditions:

$$\left\{\begin{array}{l} \rho^*(0) = -J^*(0) + 2u_l \\ \rho^*(1) = J^*(1) + 2v_r. \end{array}\right. \tag{12}$$

J^* depends on χ because $\beta \neq 0$. Moreover, it satisfies: $J^* = -\dfrac{1}{2\alpha}\dfrac{\partial \rho^*}{\partial \chi}$. The introduction of this relation in the first equation of (11) leads to the following ordinary differential equation:

$$\frac{\partial^2 \rho^*}{\partial^2 \chi} = C^2 \rho^* \ , \quad C = 2\sqrt{\alpha\beta} = h\sqrt{3\sigma_t\sigma_a}$$

whose solution is:

$$\rho^*(\chi) = ae^{\chi C} + be^{-\chi C} \ \ \{a, b\} \in \mathbb{R}$$

This statement and the second equation of (11) yield $J^*(\chi) = -\dfrac{C}{2\alpha}(ae^{\chi C} - be^{-\chi C})$.
We are now able to compute the solutions of (9):

$$\left\{\begin{array}{l} u^*(\chi) = \dfrac{a}{2}\left(1 - \dfrac{C}{2\alpha}\right)e^{\chi C} + \dfrac{b}{2}\left(1 + \dfrac{C}{2\alpha}\right)e^{-\chi C} \\ v^*(\chi) = \dfrac{a}{2}\left(1 + \dfrac{C}{2\alpha}\right)e^{\chi C} + \dfrac{b}{2}\left(1 - \dfrac{C}{2\alpha}\right)e^{-\chi C} \end{array}\right. \tag{13}$$

The boundary conditions (10) lead to a set of 2 equations with 2 unknowns $\{a, b\}$ and we get:

$$\left\{\begin{array}{l} a = 4\alpha\dfrac{(2\alpha - C)e^{-C}u_l - (2\alpha + C)v_r}{(2\alpha - C)^2 e^{-C} - (2\alpha + C)^2 e^{C}} \\ \\ b = -4\alpha\dfrac{(2\alpha + C)e^{C}u_l - (2\alpha - C)v_r}{(2\alpha - C)^2 e^{-C} - (2\alpha + C)^2 e^{C}}. \end{array}\right.$$

These identities give the following expressions of the unknown states $\hat{u}, \hat{v}$:

$$\left\{\begin{array}{l} \hat{u} = u^*(1) = \tilde{a}u_l + \tilde{b}v_r \\ \hat{v} = v^*(0) = \tilde{b}u_l + \tilde{a}v_r \end{array}\right. \tag{14}$$

with:

$$\begin{cases} \tilde{a} = \dfrac{2\sqrt{\alpha\beta}}{2\sqrt{\alpha\beta}\cosh(2\sqrt{\alpha\beta}) + (\alpha+\beta)\sinh(2\sqrt{\alpha\beta})} \\ \\ \tilde{b} = \dfrac{\alpha-\beta}{\alpha+\beta+2\sqrt{\alpha\beta}\coth(2\sqrt{\alpha\beta})}. \end{cases} \tag{15}$$

Lemma 1 *The coefficients* $\left\{\tilde{a}, \tilde{b}\right\}$ *satisfy the following properties:*

$$\begin{cases} \tilde{a} > 0 \\ \tilde{b} > 0 \\ \tilde{a} + \tilde{b} < 1. \end{cases}$$

The unknown states $\{\hat{u}, \hat{v}\}$ *are a linear combination of* $\{u_l, v_r\}$ *with positive coefficients.*

Proof Trivial because of the properties of the hyperbolic functions and $\alpha > 0$, $\beta > 0$. ◇

3.2 Godunov type scheme

If we apply the above solver to the cell C_j, with the following notations for the different states:

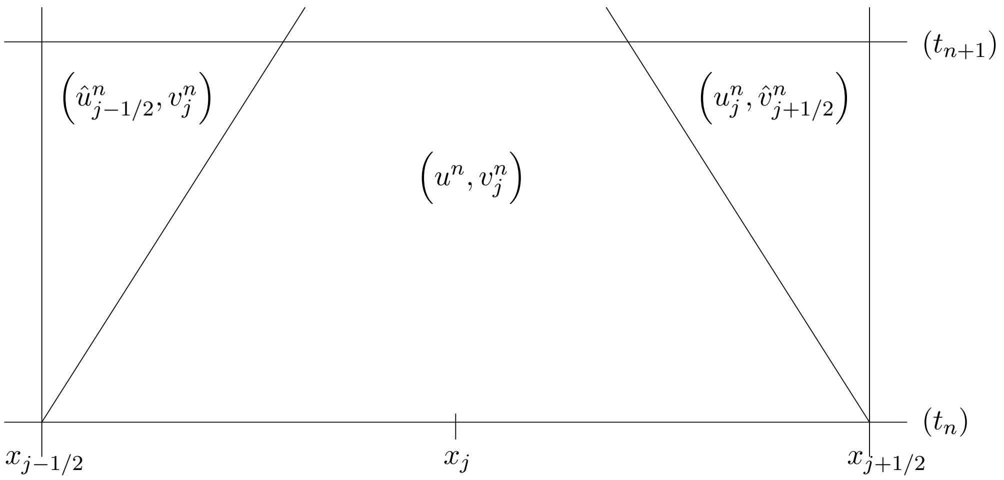

Figure 2: Riemann problem in the cell C_j.

we obtain the explicit first order Godunov type scheme:

$$\begin{cases} u^{n+1}_j = u^n_j + \dfrac{\Delta t}{h\sqrt{3}}(\hat{u}^n_{j-1/2} - u^n_j) \\ v^{n+1}_j = v^n_j + \dfrac{\Delta t}{h\sqrt{3}}(\hat{v}^n_{j+1/2} - v^n_j) \end{cases} \tag{16}$$

with:

$$\begin{cases} \hat{u}^n_{j-1/2} = \tilde{a}u^n_{j-1} + \tilde{b}v^n_j \\ \hat{v}^n_{j+1/2} = \tilde{b}u^n_j + \tilde{a}v^n_{j+1}. \end{cases} \tag{17}$$

Remark For the first and last interfaces of the mesh, the equations (9) must be slightly modified to take into account the boundary conditions:

$$\left\{\begin{array}{rcl} \dfrac{\partial u^*}{\partial \chi} & = & \dfrac{h}{2}\sqrt{3}\left(\dfrac{\sigma_t}{2}(v^*-u^*)-\dfrac{\sigma_a}{2}(u^*+v^*)\right) \\ -\dfrac{\partial v^*}{\partial \chi} & = & \dfrac{h}{2}\sqrt{3}\left(\dfrac{\sigma_t}{2}(u^*-v^*)-\dfrac{\sigma_a}{2}(u^*+v^*)\right) \end{array}\right.$$

for $\chi \in [0,1]$ and the following boundary conditions for the first interface:

$$\left\{\begin{array}{l} u^*(0)=\psi_m(\mu^+,t=0) \\ v^*(1)=v_r \end{array}\right.$$

and for the last interface:

$$\left\{\begin{array}{l} u^*(0)=u_l \\ v^*(1)=\psi_M(\mu^-,t=0). \end{array}\right.$$

$\diamond$

For the P_1 equations, the scheme reads as:

$$\left\{\begin{array}{l} \rho_j^{n+1}=\rho_j^n-\dfrac{\Delta t}{h\sqrt{3}}\tilde{a}\left(J_{j+1/2}^n-J_{j-1/2}^n\right)+\dfrac{\Delta t}{h\sqrt{3}}(\tilde{a}+\tilde{b}-1)\rho_j^n \\ J_j^{n+1}=J_j^n-\dfrac{\Delta t}{h\sqrt{3}}\tilde{a}\left(\rho_{j+1/2}^n-\rho_{j-1/2}^n\right)+\dfrac{\Delta t}{h\sqrt{3}}(\tilde{a}-\tilde{b}-1)J_j^n \end{array}\right. \tag{18}$$

where:

$$\left\{\begin{array}{l} \rho_{j+1/2}^n=\dfrac{\rho_j^n+\rho_{j+1}^n}{2}+\dfrac{J_j^n-J_{j+1}^n}{2} \\ J_{j+1/2}^n=\dfrac{J_j^n+J_{j+1}^n}{2}+\dfrac{\rho_j^n-\rho_{j+1}^n}{2}. \end{array}\right. \tag{19}$$

This relation was obtained by Buet et al. [2] in the case $\sigma_a=0$.

Because of the lemma 1, we can prove the L^∞-stability of the previous explicit schemes under the same CFL condition:

$$\frac{\Delta t}{h\sqrt{3}} \leq 1. \tag{20}$$

We will also consider the implicit following Godunov type scheme:

$$\left\{\begin{array}{l} u_j^{n+1}=u_j^n+\dfrac{\Delta t}{h\sqrt{3}}(\hat{u}_{j-1/2}^{n+1}-u_j^{n+1}) \\ v_j^{n+1}=v_j^n+\dfrac{\Delta t}{h\sqrt{3}}(\hat{v}_{j+1/2}^{n+1}-v_j^{n+1}) \end{array}\right. \tag{21}$$

where:

$$\begin{cases} \hat{u}_{j-1/2}^{n+1} = \tilde{a}u_{j-1}^{n+1} + \tilde{b}v_j^{n+1} \\ \hat{v}_{j+1/2}^{n+1} = \tilde{b}u_j^{n+1} + \tilde{a}v_{j+1}^{n+1}. \end{cases}$$

The corresponding scheme for the P_1 equations reads as:

$$\begin{cases} \rho_j^{n+1} = \rho_j^n - \dfrac{\Delta t}{h\sqrt{3}}\tilde{a}\left(J_{j+1/2}^{n+1} - J_{j-1/2}^{n+1}\right) + \dfrac{\Delta t}{h\sqrt{3}}(\tilde{a}+\tilde{b}-1)\rho_j^{n+1} \\ J_j^{n+1} = J_j^n - \dfrac{\Delta t}{h\sqrt{3}}\tilde{a}\left(\rho_{j+1/2}^{n+1} - \rho_{j-1/2}^{n+1}\right) + \dfrac{\Delta t}{h\sqrt{3}}(\tilde{a}-\tilde{b}-1)J_j^{n+1}. \end{cases} \tag{22}$$

where:

$$\begin{cases} \rho_{j-1/2}^{n+1} = \dfrac{\rho_{j-1}^{n+1} + \rho_j^{n+1}}{2} + \dfrac{J_{j-1}^{n+1} - J_j^{n+1}}{2} \\ J_{j-1/2}^{n+1} = \dfrac{J_{j-1}^{n+1} + J_j^{n+1}}{2} + \dfrac{\rho_{j-1}^{n+1} - \rho_j^{n+1}}{2}. \end{cases} \tag{23}$$

Let us note the previous implicit schemes are unconditionnally L^∞-stable.

Remark Like the schemes [2, 8] derived for the case $\sigma_a = 0$, the previous schemes satisfy the well-balanced property because the approximate solver is based on the exact solution of the stationary equations (9). ◇

Remark The schemes [2, 8] can also be recovered from the scheme (22) by defining:

$$\begin{cases} \tilde{a} = \dfrac{1}{1+\alpha} \\ \tilde{b} = \dfrac{\alpha}{1+\alpha}. \end{cases}$$

Let us note that these relations are obtained by considering that σ_a goes to 0 in the identities (15). ◇

3.3 Properties of the matrix for the implicit scheme

The above implicit scheme (21) can be rewritten as:

$$\begin{cases} -\dfrac{\Delta t}{h\sqrt{3}}\tilde{a}u_{j-1}^{n+1} + \left(1 + \dfrac{\Delta t}{h\sqrt{3}}\right)u_j^{n+1} - \dfrac{\Delta t}{h\sqrt{3}}\tilde{b}v_j^{n+1} & = u_j^n \\ -\dfrac{\Delta t}{h\sqrt{3}}\tilde{b}u_j^n + \left(1 + \dfrac{\Delta t}{h\sqrt{3}}\right)v_j^{n+1} - \dfrac{\Delta t}{h\sqrt{3}}\tilde{a}v_{j+1}^n & = v_j^n. \end{cases} \tag{24}$$

This scheme leads to the resolution of a linear system to achieve the computation of u^{n+1}, v^{n+1}. The matrix of this system satisfies some properties:

- it is non symmetric;
- if (u_j, v_j) are stored by pairs, it is is block tridiagonal;

- it is a M-matrix since $\tilde{a}+\tilde{b} \le 1$, $\tilde{a}>0$ and $\tilde{b}>0$ (we take into account the case $\sigma_a = 0$ by allowing $\tilde{a}+\tilde{b}=1$).

So, the scheme (21) is positive.
Moreover, a convenient way to solve the linear system is to use the block Gauss Seidel iteration method: its convergence is ensured by the above properties of the matrix. Let us note that the inversion of the (2,2) diagonal blocks:

$$\begin{pmatrix} 1+\dfrac{\Delta t}{h\sqrt{3}} & -\dfrac{\Delta t}{h\sqrt{3}}\tilde{b} \\ -\dfrac{\Delta t}{h\sqrt{3}}\tilde{b} & 1+\dfrac{\Delta t}{h\sqrt{3}} \end{pmatrix}$$

can be performed outside the time loop.

4 Asymptotic preserving property

We are interested in the behaviour of this numerical scheme in the diffusive regime.
To derive the scheme in the diffusive regime, the "discrete diffusive scaling":

$$\left\{ \Delta t \mapsto \frac{\Delta t}{\epsilon} \text{ , } \sigma_t \mapsto \frac{\sigma_t}{\epsilon} \right\}$$

is applied to (18). The CFL condition becomes: $\dfrac{\Delta t}{\epsilon h\sqrt{3}} \le 1$, which is very restrictive. This is why we will consider the implicit discretization (22).

Definition 1 *A numerical scheme is said to be asymptotic preserving (AP) or to have the diffusion limit if it satisfies a consistent discretization of the limiting diffusion equation (4), when ϵ goes to zero.*

The application of the "discrete diffusive scaling" to the implicit scheme (22) yields:

$$\left\{ \begin{array}{l} \rho_j^{n+1} = \rho_j^n - \dfrac{\Delta t}{\epsilon h\sqrt{3}}\tilde{a}_\epsilon \left(J_{j+1/2}^{n+1} - J_{j-1/2}^{n+1}\right) + \dfrac{\Delta t}{\epsilon h\sqrt{3}}(\tilde{a}_\epsilon + \tilde{b}_\epsilon - 1)\rho_j^{n+1} \\ J_j^{n+1} = J_j^n - \dfrac{\Delta t}{\epsilon h\sqrt{3}}\tilde{a}_\epsilon \left(\rho_{j+1/2}^{n+1} - \rho_{j-1/2}^{n+1}\right) + \dfrac{\Delta t}{\epsilon h\sqrt{3}}(\tilde{a}_\epsilon - \tilde{b}_\epsilon - 1)J_j^{n+1} \end{array} \right. \tag{25}$$

where $\{\tilde{a}_\epsilon, \tilde{b}_\epsilon\}$ denote the natural extension of $\{\tilde{a}, \tilde{b}\}$ defined by the expressions (15):

$$\left\{ \begin{array}{l} \tilde{a}_\epsilon = \dfrac{2\sqrt{\alpha_\epsilon \beta_\epsilon}}{2\sqrt{\alpha_\epsilon \beta_\epsilon}\cosh(2\sqrt{\alpha_\epsilon \beta_\epsilon}) + (\alpha_\epsilon + \beta_\epsilon)\sinh(2\sqrt{\alpha_\epsilon \beta_\epsilon})} \\ \\ \tilde{b}_\epsilon = \dfrac{\alpha_\epsilon - \beta_\epsilon}{\alpha_\epsilon + \beta_\epsilon + 2\sqrt{\alpha_\epsilon \beta_\epsilon}\coth(2\sqrt{\alpha_\epsilon \beta_\epsilon})} \end{array} \right. \tag{26}$$

with $\alpha_\epsilon = \dfrac{\alpha}{\epsilon}$ and $\beta_\epsilon = \beta\epsilon$. Let us note that $\alpha_\epsilon\beta_\epsilon$ remains constant, as the constant C, in spite of the scaling.

Proposition 1 *ρ at the order 0 verifies the following consistent discretization of the diffusion equation (4):*

$$\rho_j^{n+1,(0)} = \rho_j^{n,(0)} + \frac{\Delta t}{3\sigma_t h^2}\left(\rho_{j-1}^{n+1,(0)} - 2\rho_j^{n+1,(0)} + \rho_{j+1}^{n+1,(0)}\right) - \Delta t \sigma_a \rho_j^{n+1,(0)} + O(h^2) \tag{27}$$

Proof The relations (26) lead to:

$$\left\{\begin{array}{lcl} \dfrac{a_\epsilon}{\epsilon} & = & \dfrac{-8\alpha C}{4\alpha^2(e^{-C}-e^{C}) - 4\alpha C(e^{-C}+e^{C})\epsilon + C^2(e^{-C}-e^{C})\epsilon^2} \\ \dfrac{\tilde{a}_\epsilon + \tilde{b}_\epsilon - 1}{\epsilon} & = & \dfrac{4\alpha C(e^{-C}+e^{C}-2) + 2C^2(e^{C}-e^{-C})\epsilon}{4\alpha^2(e^{-C}-e^{C}) - 4\alpha C(e^{-C}+e^{C})\epsilon + C^2(e^{-C}-e^{C})\epsilon^2} \\ \dfrac{\tilde{a}_\epsilon - \tilde{b}_\epsilon - 1}{\epsilon} & = & -\dfrac{2\alpha(e^{C}-e^{-C}) + C(e^{C}+e^{-C}-2)\epsilon}{\alpha(e^{C}-e^{-C})\epsilon + C(e^{C}+e^{-C})\epsilon^2 + \dfrac{C^2}{4\alpha}(e^{C}-e^{-C})\epsilon^3}. \end{array}\right. \tag{28}$$

In the second equation of (25), the term $\dfrac{a_\epsilon}{\epsilon}$ being of order 0 in ϵ, there is only one term function of $\dfrac{1}{\epsilon}$: $\dfrac{-2}{\epsilon h\sqrt{3}}$ coming from $\dfrac{\tilde{a}_\epsilon - \tilde{b}_\epsilon - 1}{\epsilon h\sqrt{3}}$. This remark yields:

$$J_j^{n+1,(0)} = 0 \qquad (\forall j) \implies J_{j+1/2}^{n+1,(0)} = \frac{\rho_j^{n+1,(0)} - \rho_{j+1}^{n+1,(0)}}{2} \qquad (\forall j).$$

Because of the last statement and the identities (28), the scheme (25) on ρ_j^{n+1} rewrites as follows:

$$\begin{aligned} \rho_j^{n+1,(0)} = \rho_j^{n,(0)} & + \frac{\Delta t}{h\sqrt{3}}\left[\frac{2C}{\alpha(e^{-C}-e^{C})}\right]\left[\frac{-\rho_{j+1}^{n+1,(0)} + 2\rho_j^{n+1,(0)} - \rho_{j-1}^{n+1,(0)}}{2}\right] \\ & + \frac{\Delta t}{h\sqrt{3}}\left[\frac{C(e^{-C}+e^{C}-2)}{\alpha(e^{-C}-e^{C})}\right]\rho_j^{n+1,(0)} \end{aligned}$$

at the order 0 in ϵ.

Remember that α and C depend on h: $\alpha = h\sqrt{3}\dfrac{\sigma_t}{2}$ and $C = h\sqrt{3\sigma_t\sigma_a}$. The introduction of these definitions in the relation satisfied by $\rho_j^{n+1,(0)}$ leads to an expression depending only on C. To obtain a result free of this constant, let us make a Taylor expansion about h:

$$\begin{aligned} \rho_j^{n+1,(0)} = \rho_j^{n,(0)} & + \frac{\Delta t}{3\sigma_t}\left(\frac{\rho_{j+1}^{n+1,(0)} - 2\rho_j^{n+1,(0)} + \rho_{j-1}^{n+1,(0)}}{h^2}\right)\left(1 - \sigma_t\sigma_a h^2 + O(h^4)\right) \\ & - \Delta t\sigma_a\left(1 + \frac{1}{2}\sigma_t\sigma_a h^2 + O(h^4)\right)\rho_j^{n+1,(0)} \end{aligned}$$

which leads to the relation (27). ◇

5 Extension to a non uniform mesh and non constant (σ_a, σ_t)

If σ_a and σ_t are not constant and the mesh not uniform, to verify the WB property, one must replace (9) by:

$$\left\{\begin{array}{rcl} \dfrac{\partial u^*}{\partial \chi} & = & \dfrac{h_{j-1}+h_j}{2}\sqrt{3}\left(\dfrac{\sigma_t(\chi)}{2}(v^*-u^*)-\dfrac{\sigma_a(\chi)}{2}(u^*+v^*)\right) \\ -\dfrac{\partial v^*}{\partial \chi} & = & \dfrac{h_{j-1}+h_j}{2}\sqrt{3}\left(\dfrac{\sigma_t(\chi)}{2}(u^*-v^*)-\dfrac{\sigma_a(\chi)}{2}(u^*+v^*)\right) \end{array}\right. \tag{29}$$

where h_j satisfies $x_{j+1/2} = x_j + \dfrac{h_j}{2}$ and:

$$\left\{\begin{array}{lll} \sigma_{t,a}(\chi) = (\sigma_{t,a})_{j-1} & \text{for} & \chi < \dfrac{x_{j-1/2}-x_{j-1}}{x_j - x_{j-1}} \\ \sigma_{t,a}(\chi) = (\sigma_{t,a})_{j} & \text{for} & \chi > \dfrac{x_{j-1/2}-x_{j-1}}{x_j - x_{j-1}}. \end{array}\right.$$

The resolution of (29) can be made by introducing the variables (ρ^*, J^*) once again. In this case, we obtain a diffusion equation on ρ^* with variables coefficients:

$$\frac{\partial}{\partial \chi}\left(\frac{-1}{2\alpha}\frac{\partial \rho^*}{\partial \chi}\right) + 2\beta\rho^* = 0$$

which can easily be solved by classical techniques.

6 Numerical results

6.1 Plane source

This problem involves a purely scattering problem: $\sigma_a = 0$, $\sigma_t = 1$. The boundary conditions are : $u(x_m, \mu > 0, t) = 0$ and $u(x_M, \mu < 0, t) = 0$, with $x_m = -10$, $x_M = 10$. The initial condition is a pulse of particles located at $x = 0$:

$$\left\{\begin{array}{l} \rho(x,0) = \delta(x) \\ J(x,0) = 0 \end{array}\right.$$

In this case, the system of P_1 equations (1) has an exact solution [1]:

$$\begin{array}{rcl} \rho(x,t) & = & \dfrac{\sqrt{3}}{4}\,\sigma_t\, exp(-\dfrac{\sigma_t\, t}{2})\, I_0\,(\dfrac{\sigma_t}{2}\,\sqrt{t^2-3x^2})\, h(t-\sqrt{3}\,|x|) \\ & & +\,\dfrac{\sqrt{3}}{4}\,\sigma_t\, exp(-\dfrac{\sigma_t\, t}{2})\,\dfrac{t}{\sqrt{t^2-3x^2}}\, I_1\,(\dfrac{\sigma_t}{2}\,\sqrt{t^2-3x^2})\, h(t-\sqrt{3}\,|x|) \\ & & +\,\dfrac{\sqrt{3}}{2}\, exp(-\dfrac{\sigma_t\, t}{2})\,\delta(t-\sqrt{3}\,|x|) \end{array}$$

where h is the Heaviside function and I_1 , I_2 are Bessel functions of the first and second kind.

We compare the calculated solution with the exact solution (Fig.3) by taking $N_x = 5001$. The explicit version of the scheme was used with a CFL constant equal to 0.99. We observe a good

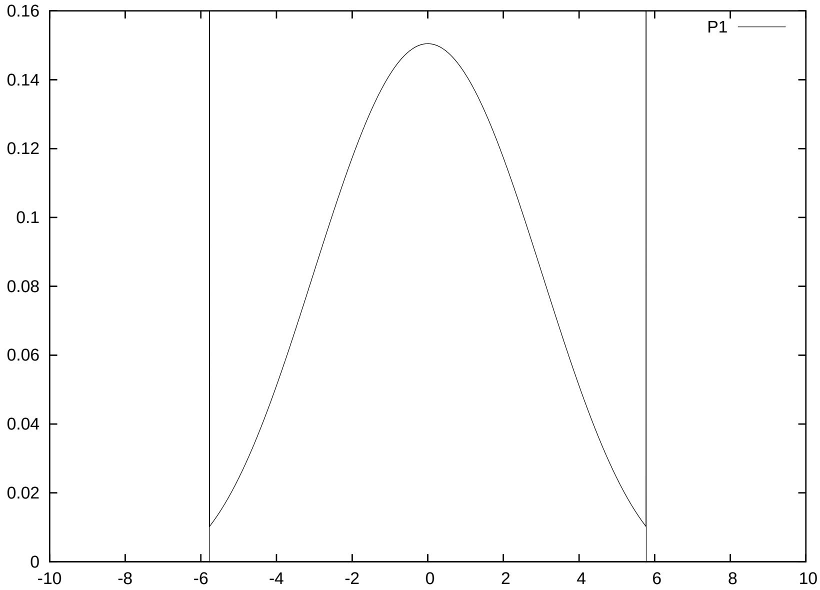

Figure 3: *Plane source - exact solution at time T=10.*

agreement between both solutions. The speed of the two opposite waves $\frac{1}{\sqrt{3}}$ and $-\frac{1}{\sqrt{3}}$ is well reproduced by the scheme (Fig.4).
The use of the implicit scheme with $\Delta t = 0.01$ (instead of $\Delta t = 0.006854$ for the explicit one) smears the solution at the position of the wave fronts, while the accuracy of the solution is preserved (Fig.5).

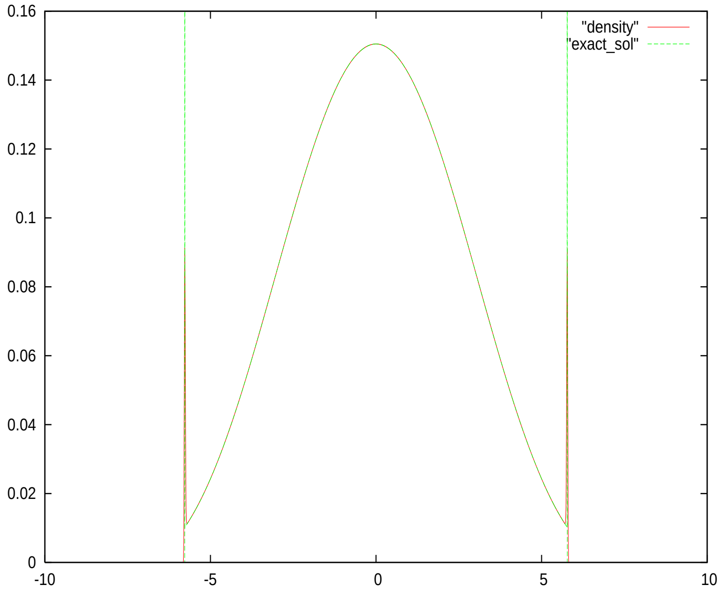

Figure 4: *Plane source with P_1 equations - comparison of the exact density and the one obtained with the explicit scheme (18).*

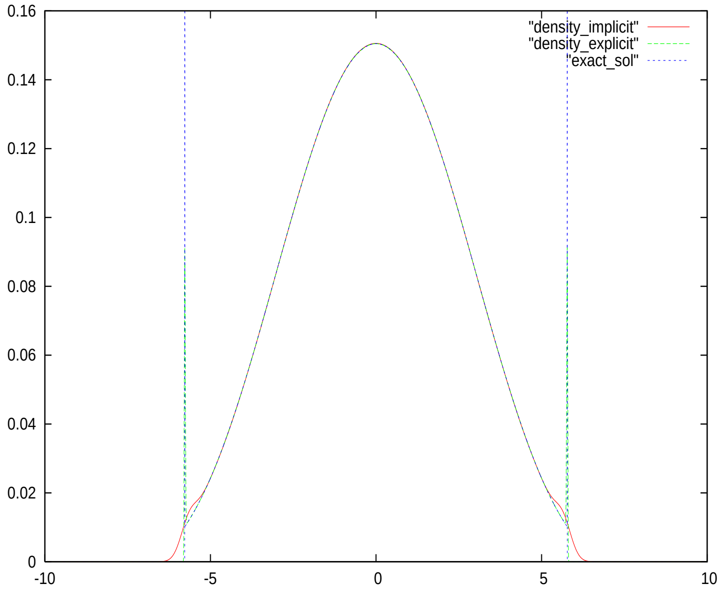

Figure 5: *Plane source with P_1 equations - comparison of the exact density and the ones obtained with the explicit and implicit scheme (18-22).*

6.2 Diffusive case

We solve the S_2 equations:

$$\left\{\begin{array}{ll} \dfrac{\partial u}{\partial t} + \dfrac{1}{\epsilon\sqrt{3}}\dfrac{\partial u}{\partial x} + \dfrac{\sigma_a}{2}(u+v) & = \dfrac{\sigma_t}{2\epsilon^2}(v-u) \\ \dfrac{\partial v}{\partial t} - \dfrac{1}{\epsilon\sqrt{3}}\dfrac{\partial v}{\partial x} + \dfrac{\sigma_a}{2}(u+v) & = \dfrac{\sigma_t}{2\epsilon^2}(u-v) \end{array}\right. \tag{30}$$

with $\sigma_a = 1$, $\sigma_t = 2$, $x_m = 0$, $x_M = 1$, $u(x_m, \mu^+, t) = \dfrac{1}{2}$ and $u(x_M, \mu^-, t) = \dfrac{1}{2}$.
When ϵ goes to zero, the stationary solution $\rho = u + v$ is solution of the diffusion equation:

$$\left\{\begin{array}{l} -\dfrac{d}{dx}\dfrac{1}{3\sigma_t}\dfrac{d\rho}{dx} + \sigma_a\rho = 0 \\ \\ \rho(0) = \rho(1) = 1 \end{array}\right.$$

which in this case is:

$$\rho = \frac{1-e^{-c}}{e^c - e^{-c}}e^{xc} + \frac{e^c - 1}{e^c - e^{-c}}e^{-xc} \tag{31}$$

with $c = \sqrt{3\sigma_a\sigma_t}$.
Due to the well-balanced property, the scheme gives the exact steady state solution when ϵ goes to zero whatever the mesh is. In this case, the steady solution is (31). This is what we observe on the figure 6 where we compare the solution of (30)with the analytical one, at the time $T = 10$, to get the stationary solution on a crude mesh $N_x = 10$ for various values of ϵ.

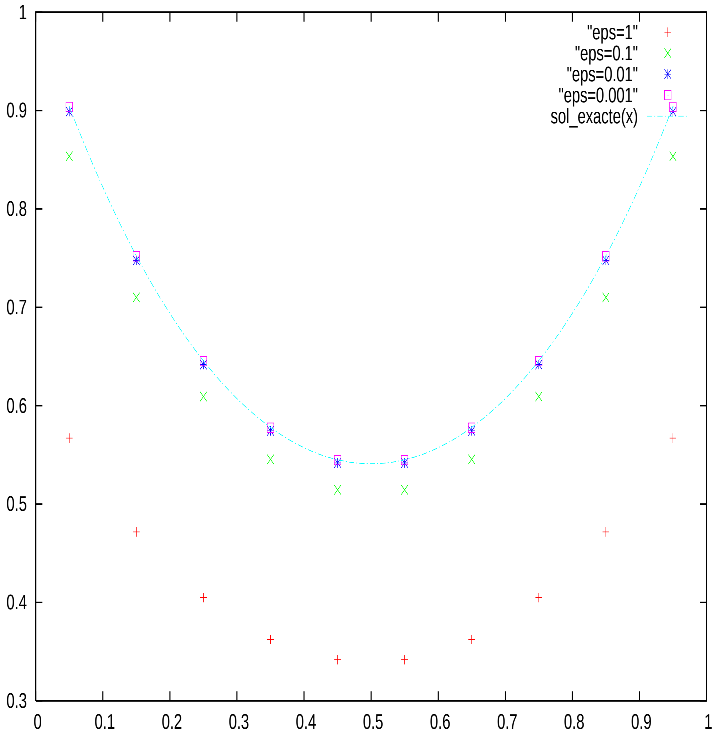

Figure 6: *Diffusive case - Gosse type scheme for the P_1 equations: convergence to the steady solution of the diffusion equation.*

7 Conclusion

In this paper, we have extended the method of L. Gosse, initially designed for the Goldstein-Taylor model, to the P_1 equations with absorption in 1D.

The resulting Godunov scheme gives the solution of the diffusion equation in the diffusive case, on a mesh resolving the diffusion scale much larger than the transport scale (asymptotic preserving property). This last result was proved by making formal expansions of the solution with respect to a small parameter representing the inverse of the number of mean free path in each cell. In the transparent scale, the scheme maintains the finite speed of propagation of the hyperbolic system. To avoid the CFL constraint on the time step which can be prohibitive in the diffusive case, the scheme has been made implicit. We have proved that the matrix of the resulting linear system is a M-matrix which ensures the positiveness of the solution. We have verified numerically that the asymptotic preserving property is satisfied.

This scheme has been extended to discretize the P_n equations in one dimension. It will be presented in a forthcoming paper.

To improve this work, we could propose an extension to second order, at least in space, of the proposed Gosse type scheme: the first order is too restrictive to compute sharp solutions, like in the plane source test for example. We are also interested in solving bidimensional problems and studying this approach on unstructured meshes.

Acknowledgments

We would like to thank Rémi Sentis whose good advices have helped us to improve this paper.

References

[1] T.A. Brunner. Riemann solvers for time-dependent transport based on the maximum entropy and spherical harmonics closures. University of Michigan: PHD, 2000.

[2] C. Buet, S. Cordier. Asymptotic preserving scheme and numerical methods for radiative hydrodynamics models. C.R. Math. Acad. Sci. Paris, Ser.I, 2004; 338.

[3] C. Buet, B. Despres. Asymptotic preserving and positive schemes for radiation hydrodynamics. J. Comput. Phys, 2006; 215: 717-740.

[4] B.G. Carlson. Transport theory : discrete ordinate quadrature over the unit sphere. Los Alamos Scientific Laboratory Report LA-4554, 1970.

[5] K.M. Case and P.F. Zweifel. Linear Transport Theory. Addison-Wesley Publishing Co., Inc. Reading, 1967.

[6] R.Mc. Clarren, J.P. Holloway, T.A. Brunner, T. Melhorn. An implicit Riemann solver for the time-dependent P_n equations. International Topical Meeting on Mathematics ans Computation, Supercomputing, Reactor Physics and Nuclear and Biological Applications, American Nuclear Society, Avignon, France, 2005.

[7] R.Mc. Clarren, J.P. Holloway, T.A. Brunner. Establishing an asymptotic diffusion limit for Riemann solvers on the time-dependent P_n equations. International Topical Meeting on Mathematics ans Computation, Supercomputing, Reactor Physics and Nuclear and Biological Applications, American Nuclear Society, Avignon, France, 2005.

[8] L. Gosse, G. Toscani. An asymptotic-preserving well-balanced scheme for the hyperbolic heat equations. C.R. Math. Acad. Sci. Paris, 2002; 334:337-342.

[9] L. Gosse, G. Toscani. Space Localization and well-balanced schemes for discrete kinetic models in diffusive regimes. SIAM J.Numer.Anal., 2003; 41:641-658.

[10] J. M. Greenberg, A. Y. Leroux. A well-balanced scheme for the numerical processing of source terms in hyperbolic equations. SIAM J. Numer. Anal., 1996; 33:1-16.